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PULSATING MAGNETOHYDRODYNAMIC FLOW SUPERPOSED ON THE STEADY LAMINAR MOTION OF AN INCOMPRESSIBLE VISCOUS FLUID IN AN ANNULAR CHANNEL

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ABSTRACT

We consider in this paper, pulsating laminar flow superposed on the steady motion of a viscous incompressible electrically conducting fluid in an annular channel between two infinitely long circular cylinders under a radially impressed magnetic field. The general magnetohydrodynamic equations are simplified by the conditions of the problem to three equations in pressure, velocity and magnetic field. One equation gives the pressure variation in the radial direction; the other two are coupled equations for the velocity and magnetic field, which are functions of the radial variable only. The solutions of these equations have been obtained in terms of Kelvin and Lommel functions and the velocity profiles are examined for different values of the frequencies of pulsation.

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Introduction

In recent years flows through small pipes have been investigated with increasing interest in the circulation system of the blood and recently theories of pulsating flow are applied to the supercharging system of reciprocating engines and the surging phenomena in power plants, and so forth (Uchida 1956).

But it would be of even greater interest to an analyst or experimentalist to investigate similar types of flow in an annular channel in the presence of magnetic fields, since the effect of impressed magnetic fields on pulsating flows are of considerable physical significance particularly in space technology problems and also biophysical problems.

Globe (1959) has considered the laminar steady state flow in an annular channel in the presence of a radial magnetic field. In the present investigation we study analytically the pulsating flow superposed on a steady flow in an annulus under a radial magnetic field. We assume in the manner of Globe that an approximation to the desired radial magnetic field may be obtained by the use of a permeable core within the annulus and a permeable cylindrical shell outside the annulus. The flux lines would close through these permeable paths at long distances from the region of interest. The source of the flux could be

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disks of permanently magnetized material between the permeable paths and the annulus channel.

The present investigation aims mainly at the analysis of the pulsating motion under such a radial magnetic field.

We now consider an infinitely long annular channel of inner radius a and outer radius b, as shown in cross section in figure 1.

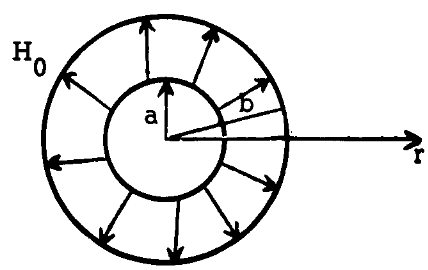

Figure 1. Annular channel with impressed radial field.

A radial magnetic field $H_0 = \frac{\alpha}{r}$, where α is a constant is impressed across the channel.

We shall now show that the independence of the electromagnetic quantities with respect to the coordinate in the direction of flow, usually assumed in plane solution, is a necessary consequence of certain conditions of the problem.

Governing Equations

On the assumptions that (1) the fluid is incompressible viscous, (2) the displacement current and free charge density are negligible, (3) the permeability and conductivity are constant scalar quantities, and (4) the Lorentz force is the only body force on the fluid, the magnetohydrodynamic equations (Cowling 1957) are

$$\nabla \times \vec{H} = \vec{J} \quad , \tag{1}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad , \tag{2}$$

$$\nabla \cdot \vec{H} = 0 \tag{3}$$

$$\vec{J} = \sigma (\vec{E} + \mu \vec{V} \times \vec{H}) \quad , \tag{4}$$

$$\nabla \cdot \vec{V} = 0 \quad , \tag{5}$$

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right] = -\nabla p + \rho \nu \nabla^2 \vec{V} + \mu \vec{J} \times \vec{H} \quad . \tag{6}$$

In these equations $\vec{H}$ is the magnetic field, $\vec{E}$ the electric field, μ the permeability, $\vec{J}$ the current density, $\vec{V}$ the velocity, p the pressure and ν the Kinematic viscosity. (M.K.S. units are understood).

We obtain now, in addition to equation (6), one more coupled equation for $\vec{H}$ and $\vec{V}$ by eliminating $\vec{J}$ and $\vec{E}$ among (1), (2) and (4) . Making use of the divergence relations (3) and (5), one obtains the following equation:

$$\frac{\partial \vec{H}}{\partial t} - (\vec{H} \cdot \nabla) \vec{V} + (\vec{V} \cdot \nabla) \vec{H} = \frac{1}{\mu \sigma} \nabla^2 \vec{H} \quad . \tag{7}$$

Equations (6) and (7) must now be expressed in cylindrical coordinates r, θ, z . The equations (6) and (7) simplify greatly because of the following conditions:

(1) We assume axial symmetry around the axis so that $\frac{\partial}{\partial\theta} = 0$.

(2) Since the flow is laminar and parallel to the axis, $v_r = v_\theta = 0$. Furthermore, from (5), $\frac{\partial v_z}{\partial z} = 0$.

(3) We assume that the applied field $H_0 = \frac{\alpha}{r}$ fixes the normal component of the magnetic field at $r = a$ and $r = b$ for all values of z, and that this is the only field impressed. Several consequences follow from this assumption:

(a) from the z-component of (4),

$$j_z = \sigma \left[E_z + \mu(\vec{V} \times \vec{H})_z \right] , \tag{8}$$

we note that j_z must vanish. This is because E_z can arise only from an applied $\vec{E}$ field or free charges in the flow, neither of which exists, and because $(\vec{V} \times \vec{H})_z$ must vanish, since $\vec{V}$ has a z-component only. Similarly from the r-component of (4) j_r vanishes, (Globe 1959). Since H_θ can arise only from an r or z component of current, there can be no θ component of $\vec{H}$ from the currents in the fluid. Since there is also no impressed magnetic field in the θ-direction, it follows that H_θ vanishes everywhere in the channel.

(b) In the manner of Globe (1959) let us consider a section of channel bounded by transverse planes at $z = \pm h$ (figure 2). Applying the divergence theorem, and Equation (3), to this volume, we get

$$\underset{\text{Volume}}{\iiint} (\nabla \cdot \vec{H})\, dQ = \underset{\text{Surface}}{\iint} \vec{H} \cdot \vec{ds} = 0 \ .$$

But

$$\iint \vec{H} \cdot \vec{ds} = -2\pi a h\, H_r(a) + 2\pi b\, H_r(b)$$

$$+ [\pi b^2 - \pi a^2][H_z(h) - H_z(-h)] \ . \qquad (10)$$

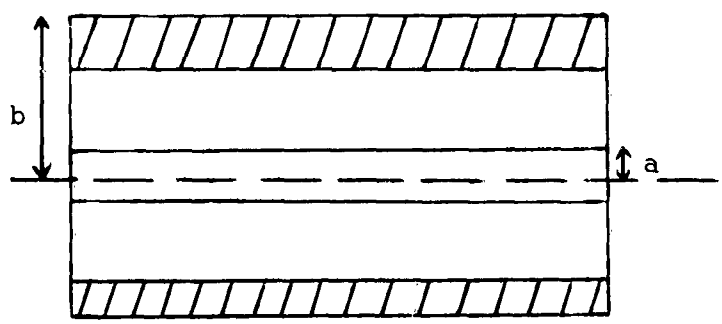

Figure 2. Section of channel bounded by transverse planes.

Since the first two terms on the right-hand side of (10) cancel each other, it follows that $H_z(h) = H_z(-h)$. Since h is arbitrary, we obtain $(\partial H_z/\partial z) = 0$. This conclusion is independent of the nature of flow.

(c) Since $\frac{\partial H_z}{\partial z} = 0$, it follows from (3) that $\frac{\partial}{\partial r}(rH_r) = 0$. The radial field must then be equal to the impressed field, and is unaffected by the flow and independent of z .

With these simplifications, the equations (6) and (7) yield:

$$\frac{\partial p}{\partial r} + \mu H_z \frac{\partial H_z}{\partial r} = 0 \quad , \tag{11}$$

$$\rho \frac{\partial v_z}{\partial t} - \frac{\mu\alpha}{r} \frac{\partial H_z}{\partial r} = -\frac{\partial p}{\partial z} + \rho\nu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right) \tag{12}$$

$$\mu\sigma \frac{\partial H_z}{\partial t} - \frac{\mu\alpha\sigma}{r} \frac{\partial v_z}{\partial r} = \frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} \tag{13}$$

H_z and v_z are functions of r and t only. It follows from (12) that $\frac{\partial p}{\partial z}$ must be independent of z . By differentiating (11) with respect to z, it can be seen that $\frac{\partial p}{\partial z}$ is independent of r also. Hence $\frac{\partial p}{\partial z}$ must be a function of time t only. We may therefore set

$$\frac{\partial p}{\partial z} = - P(t) \quad ,$$

where $P(t)$ is a function of t only.

Once H_z is determined, the variation of p across the channel may be found by integrating (11):

$$p(r, z, t) + \frac{\mu H_z^2}{2} = - P(t)z \quad .$$

Now (12) and (13) can be rewritten as:

$$\rho \frac{\partial v_z}{\partial t} - \frac{\mu\alpha}{r} \frac{\partial H_z}{\partial r} = P(t) + \rho\nu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right) \tag{14}$$

$$\mu\sigma \frac{\partial H_z}{\partial t} - \frac{\mu\alpha\sigma}{r} \frac{\partial v_z}{\partial r} = \frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} \quad . \tag{15}$$

The boundary conditions for our problem are:

$$V_z(a,t) = 0, \qquad V_z(b,t) = 0 \quad , \tag{16}$$

$$H_z(b,t) = 0 \quad , \tag{17}$$

$$\frac{\partial H_z}{\partial r}(b,t) = 0 \quad . \tag{18}$$

Equations (16) contain the no slip conditions. Equation (17) follows from the fact that $\vec{J}$ has a θ component only, so that the currents in the annular channel are like those in an infinite solenoid. These currents will therefore produce no field for $r > b$, and since there is no impressed field in the z-direction, continuity of the tangential component of $\vec{H}$ requires (17) to be true. Equation (18) is obtained as follows. We have

$$j_\theta = -\frac{\partial H_z}{\partial r} \quad .$$

But

$$j_\theta = \sigma\mu(\vec{V} \times \vec{H})_\theta \quad ,$$

and $\vec{V}$ must vanish at $r = b$. Hence j_θ must also vanish there and $\frac{\partial H_z}{\partial r}$

The Solution

Let us introduce the following non-dimensional quantities and parameters

$$\lambda = \frac{r}{a} , \quad \tau = \frac{t\nu}{a^2 (\mu\sigma\nu)^{\frac{1}{2}}} ,$$

$$V = \frac{v_z}{(\nu/a)} , \quad H = \frac{H_z}{(\nu/a)(\sigma\rho\nu)^{\frac{1}{2}}} ,$$

$$\phi(\tau) = -a^3 \rho^{-1} \nu^{-2} \frac{\partial p}{\partial z} = a^3 \rho^{-1} \nu^{-2} P(t) ,$$

$$\beta^2 = \frac{\mu^2 \alpha^2 \sigma}{\rho\nu} , \quad \gamma = (\mu\sigma\nu)^{\frac{1}{2}} ,$$

and let $\frac{b}{a} = \delta$;

β is a form of Hartmann number appropriate to an annular channel. On introducing these non-dimensional variables and parameters into the equations (14) to (18), we obtain the non-dimensional form of the latter system:

$$\frac{1}{\gamma} \frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial V}{\partial \lambda} + \frac{\beta}{\lambda} \frac{\partial H}{\partial \lambda} + \phi(\tau) \tag{19}$$

$$\gamma \frac{\partial H}{\partial \tau} = \frac{\partial^2 H}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial H}{\partial \lambda} + \frac{\beta}{\lambda} \frac{\partial V}{\partial \lambda} \tag{20}$$

$$V(\lambda, \tau) = 0 \quad \text{at} \quad \lambda = 1 \tag{21}$$

$$V(\lambda, \tau) = 0 \quad \text{at} \quad \lambda = \delta \tag{22}$$

$$H(\lambda, \tau) = 0 \quad \text{at} \quad \lambda = \delta \tag{23}$$

$$\frac{\partial H}{\partial \lambda}(\lambda, \tau) = 0 \quad \text{at} \quad \lambda = \delta . \tag{24}$$

The non-dimensional pressure gradient $\phi(\tau)$ is a function of non-dimensional time τ only and hence we can express it in the following form

$$\phi(\tau) = \phi_0 + \phi_1 e^{i\omega\tau} \tag{25}$$

where ϕ_0 and ϕ_1 are constants and may be assumed to be real for the sake of simplicity and ω also a real number denoting the frequency of the vibration.

The corresponding solutions for V and H will now be assumed as:

$$V(\lambda, \tau) = V_0(\lambda) + V_1(\lambda) e^{i\omega\tau} \tag{26}$$

$$H(\lambda, \tau) = H_0(\lambda) + H_1(\lambda) e^{i\omega\tau} \quad . \tag{27}$$

The solutions for ϕ, V and H are respectively the real parts of the expressions in (25), (26) and (27).

Substituting these forms (25) to (27) into equations (19) to (24) we have the following sets of equations:

$$\frac{d^2V_0}{d\lambda^2} + \frac{1}{\lambda}\frac{dV_0}{d\lambda} + \frac{\beta}{\lambda}\frac{dH_0}{d\lambda} + \phi_0 = 0 \quad , \tag{28}$$

$$\frac{d^2H_0}{d\lambda^2} + \frac{1}{\lambda}\frac{dH_0}{d\lambda} + \frac{\beta}{\lambda}\frac{dV_0}{d\lambda} = 0 \quad , \tag{29}$$

$$\left.\begin{aligned} &\text{and } V_0 = 0 \text{ at } \lambda = 1, \delta \\ &H_0 = 0 \text{ at } \lambda = \delta \\ &\frac{dH_0}{d\lambda} = 0 \text{ at } \lambda = \delta \end{aligned}\right\} , \tag{30}$$

$$\text{and } \frac{d^2V_1}{d\lambda^2} + \frac{1}{\lambda}\frac{dV_1}{d\lambda} + \frac{\beta}{\lambda}\frac{dH_1}{d\lambda} + \phi_1 = \frac{i\omega}{\gamma} V_1 , \tag{31}$$

$$\frac{d^2H_1}{d\lambda^2} + \frac{1}{\lambda}\frac{dH_1}{d\lambda} + \frac{\beta}{\lambda}\frac{dV_1}{d\lambda} = i\omega\gamma H_1 , \tag{32}$$

$$\left.\begin{aligned} &\text{and } V_1 = 0 \text{ at } \lambda = 1, \delta \\ &H_1 = 0 \text{ at } \lambda = \delta \\ &\frac{dH_1}{d\lambda} = 0 \text{ at } \lambda = \delta \end{aligned}\right\} . \tag{33}$$

First, we shall determine the steady part of the solution from the set (28) to (30).

We integrate (29) once and substitute for $\frac{dH_0}{d\lambda}$ into (28); then we solve the resulting second order equation for V_0; and then we substitute the resulting expression for V_0 into the first integral of (29), and finally integrate once again to get H_0. We obtain:

$$V_0(\lambda) = \frac{\phi_0}{\beta^2-4}\left[\lambda^2 - \frac{\delta^2 \sinh(\beta \ln \lambda) - \sinh(\beta \ln \lambda/\delta)}{\sinh(\beta \ln \delta)}\right] ; \; \beta^2 \neq 4 , \tag{34}$$

$$V_0(\lambda) = \frac{\phi_0}{4(\delta^4-1)}\left[\lambda^2 \ln \lambda - \delta^4 \lambda^2 \ln\frac{\lambda}{\delta} - \frac{\delta^4}{\lambda^2} \ln \delta\right] ; \; \beta^2 = 4 , \tag{35}$$

$$H_0(\lambda) = \frac{-\beta\phi_0}{\beta^2-4}\left[\frac{\lambda^2-\delta^2}{2} - \frac{\delta^2 \{\cosh(\beta\ln\lambda) - \cosh(\beta\ln\delta)\} - \cosh(\beta\ln\frac{\lambda}{\delta})+1}{\beta \sinh(\beta \ln \delta)}\right] ; \quad \beta^2 \neq 4 \tag{36}$$

$$H_0(\lambda) = \frac{\phi_0}{2(\delta^4-1)}\left[\delta^4\lambda^2 \ln\frac{\lambda}{\delta} - \frac{\delta^4}{2\lambda^2} \ln \delta - \frac{\lambda^2 \ln\lambda}{2} + \delta^2 \ln\delta + \frac{(\delta^4-1)(\delta^2-\lambda^2)}{4}\right] ; \quad \beta^2 = 4 \tag{37}$$

Next we consider the equations (31), (32) and (33). The solution of this system of equations for general values of $\gamma = (\mu\sigma\nu)^{\frac{1}{2}}$ runs into difficulties. Hence to deal with the actual physical situations, it will be sufficient to solve the system for $\gamma << 1$, since for most of the incompressible, electrically conducting viscous fluids on the surface of the earth, $\gamma << 1$.

For instance:

Fluid	$\gamma = (\mu\sigma\nu)^{\frac{1}{2}}$
Hg(20°c)	3.56×10^{-4}
Na(500°c)	1.55×10^{-3}
Pb(500°c)	0.47×10^{-3}

Hence we shall assume $\gamma << 1$ and $\omega \sim \gamma$. Then writing $k = \sqrt{\frac{\omega}{\gamma}}$, we find that the equations (31), (32) and (33) after neglecting terms of $O(\gamma^2)$, reduce to:

$$\frac{d^2H_1}{d\lambda^2} + \frac{1}{\lambda}\frac{dH_1}{d\lambda} + \frac{\beta}{\lambda}\frac{dV_1}{d\lambda} = 0 \tag{38}$$

$$\frac{d^2V_1}{d\lambda^2} + \frac{1}{\lambda}\frac{dV_1}{d\lambda} + \frac{\beta}{\lambda}\frac{dH_1}{d\lambda} + \phi_1 - ik^2 V_1 = 0 \tag{39}$$

with the conditions

$$V_1 = 0 \quad \text{at} \quad \lambda = 1, \delta$$

$$\left.\begin{array}{l} H_1 = 0 \\ \dfrac{dH_1}{d\lambda} = 0 \end{array}\right\} \text{at } \lambda = \delta \; . \tag{40}$$

From (38) after integrating once and using (40) we obtain .

$$\frac{dH_1}{d\lambda} = -\beta\frac{V_1}{\lambda} \; . \tag{41}$$

Substituting (41) into (39) we obtain

$$\lambda^2\frac{d^2V_1}{d\lambda^2} + \lambda\frac{dV_1}{d\lambda} - (ik^2\lambda^2 + \beta^2) V_1 = -\lambda^2\phi_1 \; , \tag{42}$$

which is a non-homogeneous modified Bessel equation with complex argument.

The general solution of (42) is therefore of the form

$$V_1(\lambda) = A_1\, I_\beta(k\lambda\, i^{\frac{1}{2}}) + A_2\, K_\beta(k\lambda\, i^{\frac{1}{2}}) + \frac{\phi_1}{ik^2}\, s_{1,\beta}(k\lambda\, i^{\frac{3}{2}}) \quad , \tag{43}$$

where functions I_β, K_β are modified Bessel functions of the first and second kind respectively and of order β, and $s_{1,\beta}$ is a Lommel function (Watson 1944) which occurs as a particular integral of (42). A_1 and A_2 are arbitrary constants. This particular integral can be written either as ascending or descending power series in the argument (Watson 1944) according as $1 \pm \beta \neq -(2p-1)$ or $1 \pm \beta = -(2p-1)$, where p is a positive integer. For the sake of simplicity we shall demonstrate the solution when $1 \pm \beta \neq -(2p-1)$. This condition is satisfied when $\beta =$ an odd positive integer. Thus we shall demonstrate the solution when β, the Hartmann number is an odd positive integer say equal to $2q-1$, q being positive integer.

Hence in this case we obtain the complete solution of (38) as:

$$V_1(\lambda) = -i\frac{\phi_1}{k^2}\left[\frac{\begin{array}{l} s_{1,\beta}(k\delta i^{\frac{3}{2}})\,\{K_\beta(ki^{\frac{1}{2}})I_\beta(k\lambda\, i^{\frac{1}{2}}) - I_\beta(ki^{\frac{1}{2}})\,K_\beta(k\lambda\, i^{\frac{1}{2}})\} \\ -s_{1,\beta}(ki^{\frac{3}{2}})\,\{K_\beta(k\,\delta i^{\frac{1}{2}})\,I_\beta(k\lambda\, i^{\frac{1}{2}}) - I_\beta(k\,\delta i^{\frac{1}{2}})\,K_\beta(k\lambda\, i^{\frac{1}{2}})\}\end{array}}{I_\beta(ki^{\frac{1}{2}})\,K_\beta(k\delta i^{\frac{1}{2}}) - I_\beta(k\delta i^{\frac{1}{2}})\,K_\beta(ki^{\frac{1}{2}})} + s_{1,\beta}(k\lambda\, i^{\frac{3}{2}})\right]$$

(44)

where the expression for the Lommel function is

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu+1)^2-\nu^2} - \frac{z^{\mu+3}}{\{(\mu+1)^2-\nu^2\}\{(\mu+3)^2-\nu^2\}} + \cdots\cdots\cdots$$

and $\mu \pm \nu \neq -(2p-1)$, p being positive integer.

The functions I_β and K_β can also be expressed in terms of Kelvin functions or in terms of their moduli and phases. For real x, we have

$$I_\beta(xi^{\frac{1}{2}}) = e^{-\frac{1}{2}\beta\pi i}(\mathrm{ber}_\beta x + i\,\mathrm{bei}_\beta x) = M_\beta(x)\exp\{i[\theta_\beta(x) - \tfrac{1}{2}\beta\pi]\}$$

$$K_\beta(xi^{\frac{1}{2}}) = e^{\frac{1}{2}\beta\pi i}(\mathrm{ker}_\beta x + i\,\mathrm{kei}_\beta x) = N_\beta(x)\exp\{i[\psi_\beta(x) + \tfrac{1}{2}\beta\pi]\}$$

where M_β and N_β are their respective moduli and θ_β and ψ_β are their respective phases. In an analogous manner, we shall write

$$s_{1,\beta}(xi^{\frac{3}{2}}) = P_\beta(x) + iQ_\beta(x) = L_\beta(x)\exp\{i[\chi_\beta(x)]\}$$

where $L_\beta(x)$ is the modulus and $\chi_\beta(x)$ is the phase of $s_{1,\beta}(xi^{\frac{3}{2}})$, P_β and Q_β being the real and imaginary parts of $s_{1,\beta}(xi^{\frac{3}{2}})$.

Once the solution for $V_1(\lambda)$ is known, we can now determine $H_1(\lambda)$ from (41). We have thus

$$H_1(\lambda) = \frac{i\beta\phi_1}{k^2}\left[\frac{\begin{array}{l} s_{1,\beta}(k\delta i^{\frac{3}{2}})\{K_\beta(ki^{\frac{1}{2}})[f(\lambda)-f(\delta)]-I_\beta(ki^{\frac{1}{2}})[g(\lambda)-g(\delta)]\} \\ -s_{1,\beta}(ki^{\frac{3}{2}})\{K_\beta(k\delta i^{\frac{1}{2}})[f(\lambda)-f(\delta)]-I_\beta(k\delta i^{\frac{1}{2}})[g(\lambda)-g(\delta)]\}\end{array}}{I_\beta(ki^{\frac{1}{2}})K_\beta(k\delta i^{\frac{1}{2}})-I_\beta(k\delta i^{\frac{1}{2}})K_\beta(ki^{\frac{1}{2}})} + h(\lambda)-h(\delta)\right] \tag{45}$$

where

$$f(\lambda) = \int^\lambda \frac{I_\beta(k\lambda i^{\frac{1}{2}})}{\lambda}\,d\lambda \quad ,$$

$$g(\lambda) = \int^\lambda \frac{K_\beta(k\lambda i^{\frac{1}{2}})}{\lambda}\,d\lambda \quad ,$$

$$h(\lambda) = \int^\lambda \frac{s_{1,\beta}(k\lambda i^{\frac{3}{2}})}{\lambda}\,d\lambda \quad .$$

Thus the velocity distribution and the magnetic field are now given by the real parts of (26) and (27) where $V_0(\lambda)$, $V_1(\lambda)$, $H_0(\lambda)$ and $H_1(\lambda)$ are given by (34) to (37) and (44) to (45).

We next obtain the real parts of these expressions after expressing the modified Bessel functions and Lommel functions in terms of their moduli and phases. Thus if we write the velocity distribution as

$$V(\lambda,\tau) = V_0(\lambda) + V_t(\lambda,\tau) \quad ,$$

where $V_t(\lambda,\tau)$ denotes the time-dependent part, then

$$V_t(\lambda,\tau) = \frac{\phi_1}{k^2}\left[\left\{\frac{x_1y_2-x_2y_1}{x_1^2+y_1^2} + L_\beta(k\lambda)\sin[\chi_\beta(k\lambda)]\right\}\cos\omega\tau + \left\{\frac{x_1x_2+y_1y_2}{x_1^2+y_1^2} + L_\beta(k\lambda)\cos[\chi_\beta(k\lambda)]\right\}\sin\omega\tau\right] \tag{46}$$

where

$$X_1 = \begin{vmatrix} G_{k,k\delta} & G_{k\delta,k} \\ \mathrm{Cos}\ \alpha_{k\delta,k} & \mathrm{Cos}\ \alpha_{k,k\delta} \end{vmatrix} ,$$

$$Y_1 = \begin{vmatrix} G_{k,k\delta} & G_{k\delta,k} \\ \mathrm{Sin}\ \alpha_{k\delta,k} & \mathrm{Sin}\ \alpha_{k,k\delta} \end{vmatrix} ,$$

$$X_2 = \begin{vmatrix} G_{k\lambda,k,k\delta} & G_{k,k\lambda,k\delta} \\ \mathrm{Cos}\ \alpha_{k,k\lambda,k\delta} & \mathrm{Cos}\ \alpha_{k\lambda,k,k\delta} \end{vmatrix} - \begin{vmatrix} G_{k\lambda,k\delta,k} & G_{k\delta,k\lambda,k} \\ \mathrm{Cos}\ \alpha_{k\delta,k\lambda,k} & \mathrm{Cos}\ \alpha_{k\lambda,k\delta,k} \end{vmatrix} ,$$

$$Y_2 = \begin{vmatrix} G_{k\lambda,k,k\delta} & G_{k,k\lambda,k\delta} \\ \mathrm{Sin}\ \alpha_{k,k\lambda,k\delta} & \mathrm{Sin}\ \alpha_{k\lambda,k,k\delta} \end{vmatrix} - \begin{vmatrix} G_{k\lambda,k\delta,k} & G_{k\delta,k\lambda,k} \\ \mathrm{Sin}\ \alpha_{k\delta,k\lambda,k} & \mathrm{Sin}\ \alpha_{k\lambda,k\delta,k} \end{vmatrix} ,$$

$$G_{x,y} = M_\beta(x)\ N_\beta(y)$$

$$\alpha_{x,y} = \theta_\beta(x) + \psi_\beta(y)$$

$$G_{x,y,z} = M_\beta(x)\ N_\beta(y)\ L_\beta(z)$$

and $\alpha_{x,y,z} = \theta_\beta(x)\ \psi_\beta(y)\ \chi_\beta(z)$.

$$M_\beta(x) = \sqrt{\mathrm{ber}^2_\beta x \;\; \mathrm{bei}^2_\beta x} \quad ; \quad \theta_\beta(x) = \tan^{-1} \frac{\mathrm{bei}_\beta x}{\mathrm{ber}_\beta x}$$

$$N_\beta(x) = \sqrt{\mathrm{ker}^2_\beta x + \mathrm{kei}^2_\beta x} \; ; \quad \psi_\beta(x) = \tan^{-1} \frac{\mathrm{kei}_\beta x}{\mathrm{ker}_\beta x} \quad .$$

$$L_\beta(x) = \sqrt{P^2_\beta(x) + Q^2_\beta(x)} \quad ; \quad \chi_\beta(x) = \tan^{-1} \frac{Q_\beta(x)}{P_\beta(x)} \quad .$$

and $P_\beta(x)$, $Q_\beta(x)$ being respectively the real and imaginary parts of $s_{1,\beta}(xi^{\frac{3}{2}})$ have the following expressions

$$P_\beta(x) = \frac{x^4}{(2^2-\beta^2)(4^2-\beta^2)} - \frac{x^8}{(2^2-\beta^2)(4^2-\beta^2)(6^2-\beta^2)(8^2-\beta^2)} + \dots\dots$$

$$Q_\beta(x) = -\frac{x^2}{2^2-\beta^2} + \frac{x^6}{(2^2-\beta^2)(4^2-\beta^2)(6^2-\beta^2)} - \dots\dots .$$

Total mean volumetric flow G, flowing in the z-direction is given by

$$G = \frac{1}{2\pi} \int_0^{2\pi} d\tau \int_1^{\delta} 2\pi\,\lambda\; V(\lambda,\tau)\, d\lambda \qquad (47)$$

or $$G = \frac{2\pi\phi_0}{\beta^2-4} \left[\frac{\delta^4-1}{4} + \frac{1}{\beta^2-4} \left\{ 2(\delta^4-1) - \beta(\delta^4+1)\coth(\beta \ln\delta) + 2\beta\delta^2 \operatorname{cosech}(\beta\ln\delta) \right\} \right]$$

when $\beta^2 \neq 4$;

and $$G = \frac{2\pi\phi_0}{4(\delta^4-1)} \left[\frac{(\delta^4-1)^2}{16} - \delta^4(\ln\delta)^2 \right] ,$$ when $\beta^2 = 4$.

Discussion of the Solution

We now investigate the nature of the velocity distribution $V_t(\lambda, \tau)$ and the influence of the magnetic field on it. Figures 3 and 4 give the time-dependent velocity profiles $\frac{V_t(\lambda, \tau)}{\phi_0}$ in both non-magnetic and magnetohydrodynamic flows. To illustrate the solution we have chosen $\omega = \gamma$, since $\omega \sim \gamma$, i.e., $k = 1$, radius ratio of the annulus $\frac{b}{a} = 2$ and $\phi_1 = \cdot 1 \phi_0$. The velocity profiles are obtained for different values of $\omega\tau$, viz $0°$, to $360°$. In Figure 3, the velocity profiles are obtained for the Hartmann number $\beta = 0$ and $\omega\tau = 0°$ to $360°$. In Figure 4 the velocity profiles are obtained for the Hartmann number $\beta = 5$ and $\omega\tau = 0°$ to $270°$. On comparison of Figure 3 with Figure 4, it is found that the flattening of the profile in Figure 4 which is characteristic of magnetohydrodynamic flow under a transverse field is evident. Furthermore, the magnetohydrodynamic flow profiles get even flatter and flatter as $\omega\tau$ increases from $0°$ to $90°$. Then with further increase in $\omega\tau$, the flow is reversed in direction due to pulsatory motion with the profiles becoming less and less flat between $\omega\tau = 90°$ and $180°$. With further increase in $\omega\tau$ from $180°$ to $270°$, once again the profiles get flatter and flatter. When $\omega\tau$ is between $270°$ and $360°$, the original direction of flow is restored back and after this full period of $360°$ for $\omega\tau$, once again the profiles get flatter and flatter as $\omega\tau$ begins the next cycle of values.

A more general problem of unsteady magnetohydrodynamic flow in an annular channel is presented in another paper.

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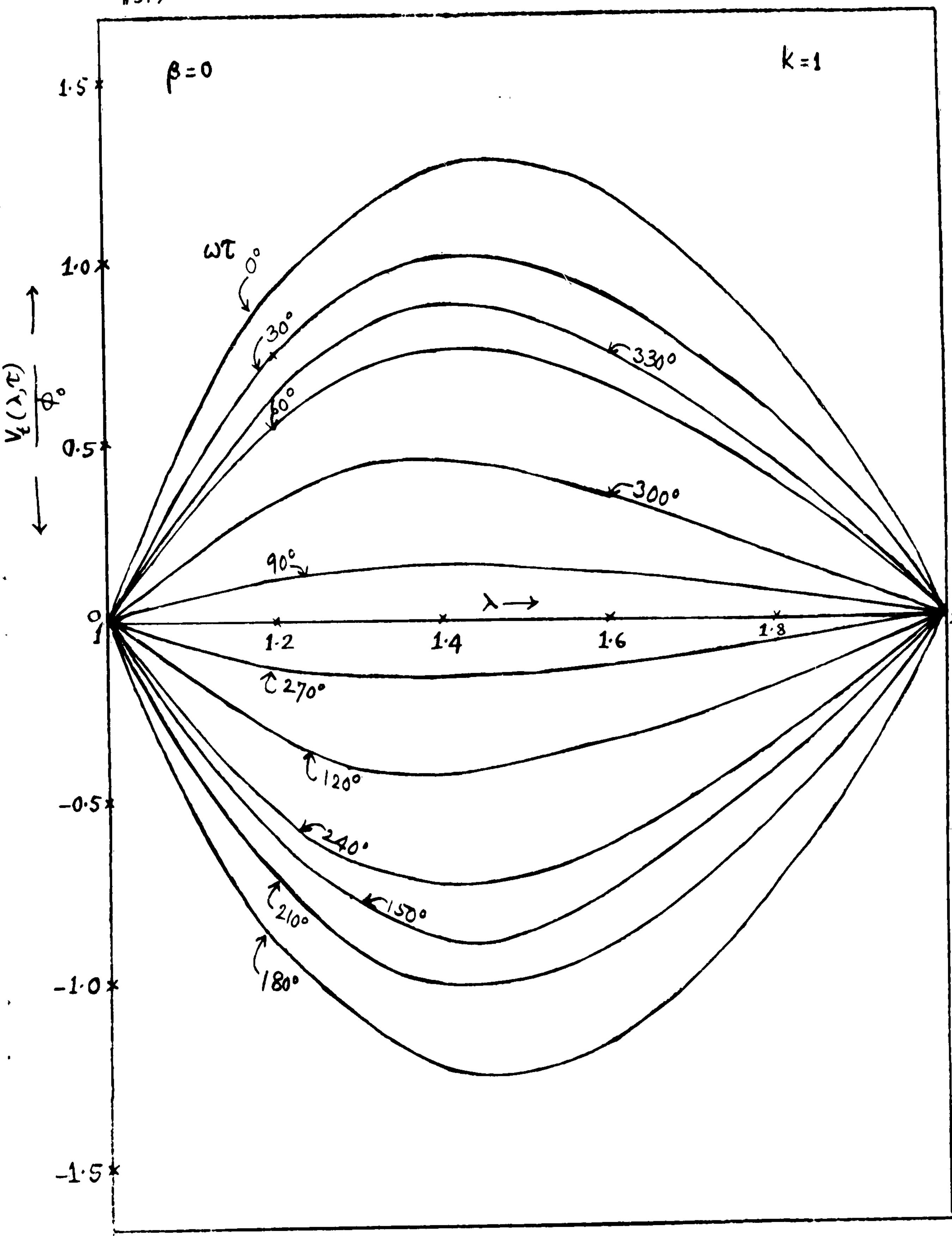
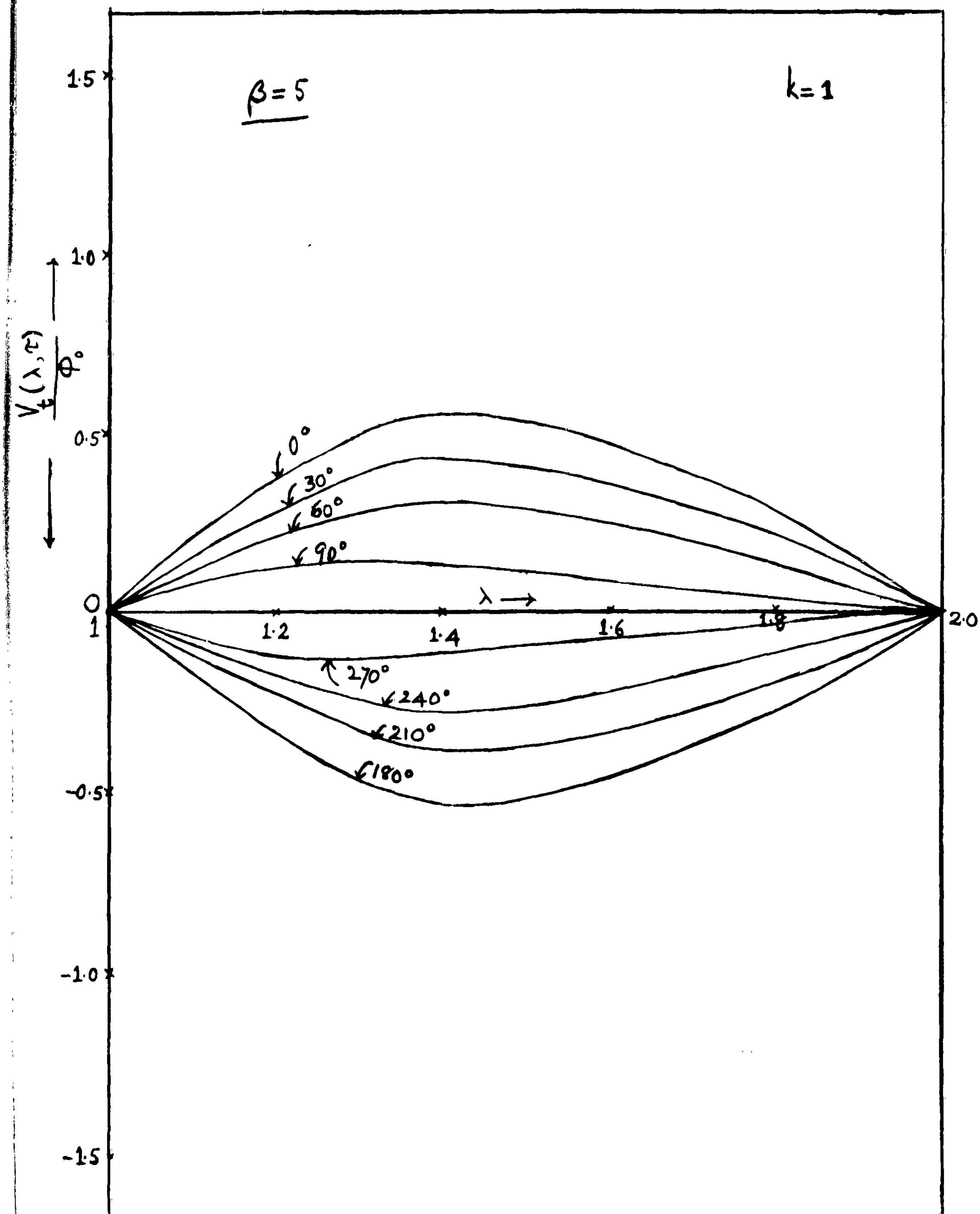

β=0
k=1
1·5
1·0
0·5
0
-0·5
-1·0
-1·5
$\frac{V_t(\lambda,\tau)}{\Phi_0}$
ωτ
0°
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
λ
1
1·2
1·4
1·6
1·8
2·0

Figure 3



β = 5
k = 1
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
V_t(λ, τ) / Φ_0
λ
1
1.2
1.4
1.6
1.8
2.0
0°
30°
60°
90°
270°
240°
210°
180°

Figure 4

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